The Chain Rule

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A special rule, **the chain rule**, exists for differentiating a function of another function. This unit illustrates this rule.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- explain what is meant by a function of a function
- state the chain rule
- differentiate a function of a function

Contents

1. Introduction

In this unit we learn how to differentiate a 'function of a function'. We first explain what is meant by this term and then learn about the Chain Rule which is the technique used to perform the differentiation.

2. A function of a function

Consider the expression $\cos x^2$. Immediately we note that this is different from the straightforward cosine function, $\cos x$. We are finding the cosine of x^2, not simply the cosine of x. We call such an expression a 'function of a function'.

Suppose, in general, that we have two functions, $f(x)$ and $g(x)$. Then

$$y = f(g(x))$$

is a function of a function. In our case, the function f is the cosine function and the function g is the square function. We could identify them more mathematically by saying that

$$f(x) = \cos x \qquad g(x) = x^2$$

so that

$$f(g(x)) = f(x^2) = \cos x^2$$

Now let's have a look at another example. Suppose this time that f is the square function and g is the cosine function. That is,

$$f(x) = x^2 \qquad g(x) = \cos x$$

then

$$f(g(x)) = f(\cos x) = (\cos x)^2$$

We often write $(\cos x)^2$ as $\cos^2 x$. So $\cos^2 x$ is also a function of a function.

In the following section we learn how to differentiate such a function.

3. The chain rule

In order to differentiate a function of a function, $y = f(g(x))$, that is to find $\frac{\mathrm{d}y}{\mathrm{d}x}$, we need to do two things:

1. Substitute $u = g(x)$. This gives us

$$y = f(u)$$

Next we need to use a formula that is known as the Chain Rule.

2. **Chain Rule**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

Key Point

Chain rule:

To differentiate $y = f(g(x))$, let $u = g(x)$. Then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example

Suppose we want to differentiate $y = \cos x^2$.

Let $u = x^2$ so that $y = \cos u$.

It follows immediately that

$$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = -\sin u$$

The chain rule says

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

and so

$$\begin{aligned} \frac{dy}{dx} &= -\sin u \times 2x \\ &= -2x \sin x^2 \end{aligned}$$

Example

Suppose we want to differentiate $y = \cos^2 x = (\cos x)^2$.

Let $u = \cos x$ so that $y = u^2$

It follows that

$$\frac{du}{dx} = -\sin x \qquad \frac{dy}{du} = 2u$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2u \times -\sin x \\ &= -2 \cos x \sin x \end{aligned}$$

Example

Suppose we wish to differentiate $y = (2x - 5)^{10}$.

Now it might be tempting to say 'surely we could just multiply out the brackets'. To multiply out the brackets would take a long time and there are lots of opportunities for making mistakes. So let us treat this as a function of a function.

Let $u = 2x - 5$ so that $y = u^{10}$. It follows that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}u} = 10u^9$$

Then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 10u^9 \times 2 \\ &= 20(2x-5)^9\end{aligned}$$

4. Some examples involving trigonometric functions

In this section we consider a trigonometric example and develop it further to a more general case.

Example

Suppose we wish to differentiate $y = \sin 5x$.

Let $u = 5x$ so that $y = \sin u$. Differentiating

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 5 \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$$

From the chain rule

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \cos u \times 5 \\ &= 5\cos 5x\end{aligned}$$

Notice how the 5 has appeared at the front, - and it does so because the derivative of $5x$ was 5. So the question is, could we do this with any number that appeared in front of the x, be it 5 or 6 or $\frac{1}{2}$, 0.5 or for that matter n ?

So let's have a look at another example.

Example

Suppose we want to differentiate $y = \sin nx$.

Let $u = nx$ so that $y = \sin u$. Differentiating

$$\frac{\mathrm{d}u}{\mathrm{d}x} = n \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$$

Quoting the formula again:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

So

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos u \times n \\ &= n\cos nx\end{aligned}$$

So the n's have behaved in exactly the same way that the 5's behaved in the previous example.

Key Point

if $y = \sin nx$ then $\dfrac{dy}{dx} = n\cos nx$

For example, suppose $y = \sin 6x$ then $\dfrac{dy}{dx} = 6\cos 6x$ just by using the standard result.

Similar results follow by differentiating the cosine function:

Key Point

if $y = \cos nx$ then $\dfrac{dy}{dx} = -n\sin nx$

So, for example, if $y = \cos\frac{1}{2}x$ then $\dfrac{dy}{dx} = -\frac{1}{2}\sin\frac{1}{2}x$.

5. A simple technique for differentiating directly

In this section we develop, through examples, a further result.

Example

Suppose we want to differentiate $y = e^{x^3}$.

Let $u = x^3$ so that $y = e^u$. Differentiating

$$\frac{du}{dx} = 3x^2 \qquad \frac{dy}{du} = e^u$$

Quoting the formula again:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

So

$$\begin{aligned}\frac{dy}{dx} &= e^u \times 3x^2 \\ &= 3x^2 e^{x^3}\end{aligned}$$

We will now explore how this relates to a general case, that of differentiating $y = f(g(x))$.
To differentiate $y = f(g(x))$, we let $u = g(x)$ so that $y = f(u)$.

The chain rule states

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

In what follows it will be convenient to reverse the order of the terms on the right:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}y}{\mathrm{d}u}$$

which, in terms of f and g we can write as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{d}{dx}\left(g(x)\right) \times \frac{d}{du}\left(f(g((x)))\right)$$

This gives us a simple technique which, with some practice, enables us to apply the chain rule directly

Key Point

(i) given $y = f(g(x))$, identify the functions $f(u)$ and $g(x)$ where $u = g(x)$.

(ii) differentiate g and multiply by the derivative of f
where it is understood that the argument of f is $u = g(x)$.

Example

To differentiate $y = \tan x^2$ we apply these two stages:

(i) first identify $f(u)$ and $g(x)$: $f(u) = \tan u$ and $g(x) = x^2$.

(ii) differentiate $g(x)$: $\frac{\mathrm{d}g}{\mathrm{d}x} = 2x$. Multiply by the derivative of $f(u)$, which is $\sec^2 u$ to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \sec^2 x^2$$

Example

To differentiate $y = \mathrm{e}^{1+x^2}$.

(i) first identify $f(u)$ and $g(x)$: $f(u) = \mathrm{e}^u$ and $g(x) = 1 + x^2$.

(ii) differentiate $g(x)$: $\frac{\mathrm{d}g}{\mathrm{d}x} = 2x$. Multiply by the derivative of $f(u)$, which is e^u to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\,\mathrm{e}^{1+x^2}$$

You should be able to verify the remaining examples purely by inspection. Try it!

Example

$$\begin{aligned} y &= \sin(x + e^x) \\ \frac{dy}{dx} &= (1 + e^x)\cos(x + e^x) \end{aligned}$$

Example

$$\begin{aligned} y &= \tan(x^2 + \sin x) \\ \frac{dy}{dx} &= (2x + \cos x).\sec^2(x^2 + \sin x) \end{aligned}$$

Example

$$\begin{aligned} y &= (2 - x^5)^9 \\ \frac{dy}{dx} &= -5x^4.\ 9(2 - x^5)^8 \\ &= -45x^4(2 - x^5)^8 \end{aligned}$$

Example

$$\begin{aligned} y &= \ln(x + \sin x) \\ \frac{dy}{dx} &= (1 + \cos x)\,.\,\frac{1}{x + \sin x} \\ &= \frac{1 + \cos x}{x + \sin x} \end{aligned}$$

Exercises

1. Find the derivative of each of the following:
   a) $(3x - 7)^{12}$ b) $\sin(5x + 2)$ c) $\ln(2x - 1)$ d) e^{2-3x}
   e) $\sqrt{5x - 3}$ f) $(6x + 5)^{5/3}$ g) $\dfrac{1}{(3 - x)^4}$ h) $\cos(1 - 4x)$
2. Find the derivative of each of the following:
   a) $\ln(\sin x)$ b) $\sin(\ln x)$ c) $e^{-\cos x}$ d) $\cos(e^{-x})$
   e) $(\sin x + \cos x)^3$ f) $\sqrt{1 + x^2}$ g) $\dfrac{1}{\cos x}$ h) $\dfrac{1}{x^2 + 2x + 1}$

3. Find the derivative of each of the following:

a) $\ln\left(\sin^2 x\right)$ b) $\sin^2(\ln x)$ c) $\sqrt{\cos(3x-1)}$ d) $\left[1+\cos\left(x^2-1\right)\right]^{3/2}$

Answers

1. a) $36(3x-7)^{11}$ b) $5\cos(5x+2)$ c) $\dfrac{2}{2x-1}$ d) $-3\mathrm{e}^{2-3x}$

   e) $\dfrac{5}{2\sqrt{5x-3}}$ f) $10(6x+5)^{2/3}$ g) $\dfrac{4}{(3-x)^5}$ h) $4\sin(1-4x)$

2. a) $\dfrac{\cos x}{\sin x}=\cot x$ b) $\dfrac{\cos(\ln x)}{x}$ c) $\sin x\mathrm{e}^{-\cos x}$

   d) $\mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right)$ e) $3(\cos x-\sin x)(\sin x+\cos x)^2$ f) $\dfrac{x}{\sqrt{1+x^2}}$

   g) $\dfrac{\sin x}{\cos^2 x}=\tan x\sec x$ h) $\dfrac{-2(x+1)}{(x^2+2x+1)^4}=\dfrac{-2}{(x+1)^3}$

3. a) $\dfrac{2\cos x}{\sin x}=2\cot x$ b) $\dfrac{2\sin(\ln x)\cos(\ln x)}{x}$

   c) $\dfrac{-3\sin(3x-1)}{2\sqrt{\cos(3x-1)}}$ d) $-3x\sin\left(x^2-1\right)\left[1+\cos\left(x^2-1\right)\right]^{1/2}$

