mathcentre

Differentiation of the sine and cosine functions from first principles

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In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- differentiate the function $\sin x$ from first principles
- differentiate the function $\cos x$ from first principles

Contents

1. Introduction

In this unit we look at how to differentiate the functions $f(x) = \sin x$ and $f(x) = \cos x$ from first principles. We need to remind ourselves of some familiar results.

The derivative of $f(x)$.

The definition of the derivative of a function $y = f(x)$ is

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Two trigonometric identities.

We will make use of the trigonometric identities

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

The limit of the function $\dfrac{\sin\theta}{\theta}$.

As θ (measured in radians) approaches zero, the function $\dfrac{\sin\theta}{\theta}$ tends to 1. We write this as

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

This result can be justified by choosing values of θ closer and closer to zero and examining the behaviour of $\dfrac{\sin\theta}{\theta}$.

Table 1 shows values of θ and $\dfrac{\sin\theta}{\theta}$ as θ becomes smaller.

θ	$\sin\theta$	$\frac{\sin\theta}{\theta}$
1	0.84147	0.84147
0.1	0.09983	0.99833
0.01	0.00999	0.99983

Table 1: The value of $\frac{\sin\theta}{\theta}$ as θ tends to zero is 1.

You should verify these results with your calculator to appreciate that the value of $\dfrac{\sin\theta}{\theta}$ approaches 1 as θ tends to zero.

We now use these results in order to differentiate $f(x) = \sin x$ from first principles.

2. Differentiating $f(x) = \sin x$

Here $f(x) = \sin x$ so that $f(x + \delta x) = \sin(x + \delta x)$.
So

$$f(x + \delta x) - f(x) = \sin(x + \delta x) - \sin x$$

The right hand side is the difference of two sine terms. We use the first trigonometric identity (above) to write this in an alternative form.

$$\begin{aligned}
\sin(x + \delta x) - \sin x &= 2 \cos \frac{x + \delta x + x}{2} \sin \frac{\delta x}{2} \\
&= 2 \cos \frac{2x + \delta x}{2} \sin \frac{\delta x}{2} \\
&= 2 \cos(x + \frac{\delta x}{2}) \sin \frac{\delta x}{2}
\end{aligned}$$

Then, using the definition of the derivative

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \\
&= \frac{2 \cos(x + \frac{\delta x}{2}) \sin \frac{\delta x}{2}}{\delta x}
\end{aligned}$$

The factor of 2 can be moved into the denominator as follows, in order to write this in an alternative form:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\cos(x + \frac{\delta x}{2}) \sin \frac{\delta x}{2}}{\delta x / 2} \\
&= \cos\left(x + \frac{\delta x}{2}\right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}
\end{aligned}$$

We now let δx tend to zero. Consider the term $\frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$ and use the result that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ with $\theta = \frac{\delta x}{2}$. We see that

$$\lim_{\delta x \to 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} = 1$$

Further,

$$\lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) = \cos x$$

So finally,

$$\frac{dy}{dx} = \cos x$$

3. The derivative of $f(x) = \cos x$.

Here $f(x) = \cos x$ so that $f(x + \delta x) = \cos(x + \delta x)$.
So

$$f(x + \delta x) - f(x) = \cos(x + \delta x) - \cos x$$

The right hand side is the difference of two cosine terms. This time we use the trigonometric identity

$$\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

to write this in an alternative form.

$$\cos(x + \delta x) - \cos x = -2 \sin \frac{x + \delta x + x}{2} \sin \frac{\delta x}{2}$$

$$-2 \sin \frac{2x + \delta x}{2} \sin \frac{\delta x}{2}$$

$$= -2 \sin(x + \frac{\delta x}{2}) \sin \frac{\delta x}{2}$$

Then, using the definition of the derivative

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \\ &= \frac{-2 \sin(x + \frac{\delta x}{2}) \sin \frac{\delta x}{2}}{\delta x} \end{aligned}$$

The factor of 2 can be moved as before, in order to write this in an alternative form:

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\sin(x + \frac{\delta x}{2}) \sin \frac{\delta x}{2}}{\delta x/2} \\ &= -\sin\left(x + \frac{\delta x}{2}\right) \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} \end{aligned}$$

We now want to let δx tend to zero. As before

$$\lim_{\delta x \to 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} = 1$$

Further,

$$\lim_{\delta x \to 0} -\sin\left(x + \frac{\delta x}{2}\right) = -\sin x$$

So finally,

$$\frac{dy}{dx} = -\sin x$$

So, we have used differentiation from first principles to find the derivatives of the functions $\sin x$ and $\cos x$.

