Highest Common Factor, Lowest Common Multiple

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The *highest common factor* (HCF) of two positive integers (whole numbers) is the largest number which divides them both. Their *lowest common multiple* (LCM) is the smallest number which both of them divide. As we will see later, the HCF of 648 and 936 is 72, their LCM is 8424.

Exercise

Using a calculator, check that 648 and 936 are both divisible by 72 and that both divide 8424.

Prime numbers

A positive integer is a *prime number*, or simply a prime, if it has no other divisors other than 1 and itself; 1 is not regarded as a prime. The primes less than 100 are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

There is no known way of telling if a given positive integer n is a prime other than checking whether it is divisible by any of the primes up to $\sqrt{n}+1$. It is only necessary to go as far as $\sqrt{n}+1$ because if $n = a \times b$ then one of a and b is smaller than $\sqrt{n}+1$. *Titanic primes* — ones with more than 1,000 decimal digits — have important applications in the secure encryption of information. Many thousands have been found using highly sophisticated mathematical and computational methods.

Exercise

Use a calculator to show that 8629 is a prime by checking that it is not divisible by any of the primes in the list above. Note that $\sqrt{8629} \approx 92.9$. Is 8633 a prime?

Prime factorisation

Every positive integer other than 1 can be broken into a product of primes in a unique way, called its *prime factorisation*. Trial and error is the only known way of finding the prime factorisation.

Example

740	$=$	2×370	740 is even so divisible by 2
	$=$	$2 \times 2 \times 185$	370 is even so divisible by 2
			$185/3$ is not a whole number so 185 is not divisible by 3
	$=$	$2 \times 2 \times 5 \times 37$	185 is divisible by 5

Since 37 is a prime no further breaking into factors is possible. The prime factorisation of 740 is $2 \times 2 \times 5 \times 37$, sometimes written $2^2 \times 5 \times 37$.

Exercise

Show that the prime factorisation of 648 is $2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3$ and the prime factorisation of 936 is $2 \times 2 \times 2 \times 3 \times 3 \times 13$

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Using the prime factorisation to find the HCF

Once the prime factorisations of two numbers have been found then their HCF is equal to the overlap between the prime factors.

Example

$$648 = \underbrace{2 \times 2 \times 2 \times 3 \times 3}_{\text{the overlap}} \times 3 \times 3 \quad \text{and} \quad 936 = \underbrace{2 \times 2 \times 2 \times 3 \times 3}_{\text{the overlap}} \times 13$$

so $\text{HCF}(648,\, 936) = 2 \times 2 \times 2 \times 3 \times 3 = 72$.

Exercise

Find the prime factorisation of 884 and show that $\text{HCF}(884, 936) = 52$.

Using the prime factorisation to find the LCM

Once the prime factorisations of two numbers have been found then their LCM is equal to the product of prime factors from one number or the other without repeating the overlap.

Example

$$\text{LCM}(648,\, 936) = \underbrace{2 \times 2 \times 2 \times 3 \times 3}_{\text{the overlap}} \ \underbrace{\times 3 \times 3}_{\text{from 648}} \quad \underbrace{\times 13}_{\text{from 936}} \quad = 8424$$

This also illustrates a general rule

$$\text{LCM}(a, b) = \frac{a \times b}{\text{HCF}(a, b)} \tag{1}$$

Euclid's algorithm to find the HCF

This is a way of finding the HCF of two positive integers a and b without the need to find their prime factorisations. Suppose $a > b$. Step 1 is to find the remainder r when a is divided by b. Step 2 is to get a new value of a by putting it equal to the old value of b and a new value of b by putting it equal to r. Repeat steps 1 and 2 until $r = 0$; $\text{HCF}(a, b)$ is equal to the final value of b.

Example

The steps to find $\text{HCF}(970, 189)$ and $\text{HCF}(1620, 228)$ using Euclid's algorithm are:

$$\begin{array}{lrcl@{\qquad}rcl}
\text{Step 1:} & 970 & = & 189 \times 5 + 25 & 1620 & = & 228 \times 7 + 24 \\
\text{Step 2:} & & \swarrow & \swarrow & & \swarrow & \swarrow \\
 & 189 & = & 25 \times 7 + 14 & 228 & = & 24 \times 9 + 12 \\
 & & \swarrow & \swarrow & & \swarrow & \swarrow \\
 & 25 & = & 14 \times 1 + 11 & 24 & = & \boxed{12} \times 2 + 0 \\
 & & \swarrow & \swarrow & & & \\
 & 14 & = & 11 \times 1 + 3 & & & \\
 & & \swarrow & \swarrow & & & \\
 & 11 & = & 3 \times 3 + 2 & & & \\
 & & \swarrow & \swarrow & & & \\
 & 3 & = & 2 \times 1 + 1 & & & \\
 & & \swarrow & \swarrow & & & \\
 & 2 & = & \boxed{1} \times 2 + 0 & & &
\end{array}$$

so $\text{HCF}(970, 189) = 1$ and $\text{HCF}(1620, 228) = 12$. 970 and 189 are *coprime* because their HCF$= 1$.

Exercise

Use equation (1) to find $\text{LCM}(1620, 228)$.

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